

Canonical Analysis of Unimodular Gravity

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ABSTRACT: This short note is devoted to the Hamiltonian analysis of the Unimodular Gravity. We treat the unimodular gravity as General Relativity action with the unimodular constraint imposed with the help of Lagrange multiplier. We perform the canonical analysis of the resulting theory and determine its constraint structure.

KEYWORDS: Hamiltonian Formalism, Unimodular Gravity .

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1. Introduction and Summary

Unimodular gravity is obtained from Einstein-Hilbert action in which the unimodular condition

$$\sqrt{-\det \hat{g}_{\mu\nu}} = 1 \quad (1.1)$$

is imposed from the beginning [1, 2]. The resulting field equations correspond to the traceless Einstein equations and can be shown that they are equivalent to the full Einstein equations with the cosmological constant term Λ , where Λ enters as an integration constant. In other words we see clear equivalence between unimodular gravity and general relativity. On the other hand the idea that the cosmological constant arises as an integration constant is very attractive and it is one of the motivation for the study of the unimodular gravity, for recent study, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

The fact that the determinant of the metric is fixed has clearly profound consequences on the structure of given theory. First of all it reduces the full group of diffeomorphism to invariance under the group of unimodular general coordinate transformations which are transformations that leave the determinant of the metric unchanged. Further, the fact that the metric is fixed could have important consequences for the Hamiltonian formulation of given theory. Some aspects of the Hamiltonian treatment of unimodular gravity were analyzed in [15, 16]. Then very important contribution to this analysis was presented in [17], where the condition (1.1) was fixed by hand from the beginning. On the other hand we mean that it would be desirable to impose this condition using the Lagrange multiplier term that is added to the gravity action. In fact, similar analysis was performed in [18] using very elegant formalism of geometrodynamics [19, 20] which is manifestly diffeomorphism invariant. However this elegant formulation can be achieved with the help of the introducing of the collection of the scalar fields which on the other hand makes the analysis more complicated. Our goal is to perform the Hamiltonian analysis in more straightforward manner when we consider general relativity action where the constraint (1.1) is imposed with the help of the Lagrange multiplier. Clearly this expression breaks the diffeomorphism invariance explicitly and we would like to see the consequence of the presence of this term on the Hamiltonian structure of given theory. It turns out

that given structure is rather interesting. Explicitly, we consider Lagrange multiplier as the dynamical variable where its momentum is the primary constraint of the theory. We also find that the momentum conjugate to the lapse N is not the first class constraint but together with (1.1) form the collection of the second class constraints. Then we find another set of constraints that implies that the Lagrange multiplier has to depend on time only. Finally we split the Hamiltonian constraints into collection of $\infty^3 - 1$ constraints (in terminology of [18]) and one constraint that together with the momentum conjugate to the zero mode part of the Lagrange multiplier forms the second class constraints. This is subtle difference with respect to the case of general relativity that possesses $4\infty^3$ first class constraints. On the other hand the presence of the global constraint that relates the dynamical gravity fields and embedding fields was mentioned in [18] and we mean that our result has closed overlap with the conclusion derived there.

As the next step we perform the Hamiltonian analysis of the unimodular theory proposed in [17]. Now due to the fact that given theory is manifestly covariant the analysis is more straightforward and we derive $4\infty^3$ first class constraints. On the other hand the structure of the Hamiltonian constraint is different from the Hamiltonian constraint of the general relativity since now it contains the term corresponding the momentum conjugate to time component of the vector field \mathcal{F}^μ . Now due to the fact that the Hamiltonian does not depend on this field explicitly we find that this momentum is constant on shell and hence its constant value can be considered as an effective cosmological constant.

This paper is organized as follows. In the next section (2) we perform the Hamiltonian analysis of unimodular theory with constraint (1.1) included into the action using the Lagrange multiplier. Then in section (3) we perform the Hamiltonian analysis of the formulation of unimodular gravity proposed in [17].

2. Hamiltonian Analysis of Unimodular Gravity

In this section we perform the Hamiltonian analysis of the unimodular gravity where the condition (1.1) is imposed using the Lagrange multiplier term included into the action. Explicitly, we consider the action

$$S = \frac{1}{16\pi G} \int d^4x (\sqrt{-\hat{g}}^{(4)} R[\hat{g}] - \Lambda(\sqrt{-\hat{g}} - 1)) , \quad (2.1)$$

where $^{(4)}R$ is four dimensional curvature and where $\Lambda(x)$ is Lagrange multiplier.

To proceed to the canonical formulation we use the well know $3 + 1$ formalism that is the fundamental ingredient of the Hamiltonian formalism of any theory of gravity ¹. We consider $3 + 1$ dimensional manifold \mathcal{M} with the coordinates x^μ , $\mu = 0, \dots, 3$ and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, x^2, x^3)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, +, +)$. Suppose that \mathcal{M} can be foliated by a family of space-like surfaces Σ defined by $t = x^0$. Let g_{ij} , $i, j = 1, 2, 3$ denotes the metric on Σ with inverse

¹For recent review, see [21].

g^{ij} so that $g_{ij}g^{jk} = \delta_i^k$. We further introduce the operator ∇_i that is covariant derivative defined with the metric g_{ij} . We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

$$\begin{aligned}\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}_{ij} = g_{ij}, \\ \hat{g}^{00} &= -\frac{1}{N^2}, \quad \hat{g}^{0i} = \frac{N^i}{N^2}, \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2}.\end{aligned}\tag{2.2}$$

Then the standard canonical analysis leads to the bare Hamiltonian in the form

$$H = \int d^3\mathbf{x} (N\mathcal{H}_T + N^i \mathcal{H}_i + \Omega(\sqrt{g}N - 1) + v_N \pi_N + v_i \pi_i + v_\Omega p_\Omega), \tag{2.3}$$

where

$$\mathcal{H}_T = \frac{16\pi G}{\sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - \frac{\sqrt{g}}{16\pi G} R, \quad \mathcal{H}_i = -2g_{ik} \nabla_j \pi^{jk}, \tag{2.4}$$

where

$$\mathcal{G}_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk}) - \frac{1}{2}g_{ij}g_{kl}, \tag{2.5}$$

and where R is three dimensional curvature. Further, π^{ij} are momenta conjugate to g_{ij} with non-zero Poisson bracket

$$\{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}). \tag{2.6}$$

Finally, $\pi_N \approx 0$, $\pi_i \approx 0$, $p_\Omega \approx 0$ are primary constraints where π_N, π_i, p_Ω are momenta conjugate to N, N^i and Ω respectively with non-zero Poisson brackets

$$\{N(\mathbf{x}), \pi_N(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad \{N^i(\mathbf{x}), \pi_j(\mathbf{y})\} = \delta_j^i \delta(\mathbf{x} - \mathbf{y}), \quad \{\Omega(\mathbf{x}), p_\Omega(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}). \tag{2.7}$$

It is also useful to introduce the smeared form of the constraints $\mathcal{H}_T, \mathcal{H}_i$

$$\mathbf{T}_T(X) = \int d^3\mathbf{x} X \mathcal{H}_T, \quad \mathbf{T}_S(X^i) = \int d^3\mathbf{x} X^i \mathcal{H}_i, \tag{2.8}$$

where X, X^i are functions on Σ . For further purposes we also introduce the well known Poisson brackets

$$\begin{aligned}\{\mathbf{T}_T(X), \mathbf{T}_T(Y)\} &= \mathbf{T}_S((X\partial_i Y - Y\partial_i X)g^{ij}), \\ \{\mathbf{T}_S(X), \mathbf{T}_T(Y)\} &= \mathbf{T}_T(X^i \partial_i Y), \\ \{\mathbf{T}_S(X^i), \mathbf{T}_S(Y^j)\} &= \mathbf{T}_S(X^j \partial_j Y^i - Y^j \partial_j X^i).\end{aligned}\tag{2.9}$$

Now we proceed to the analysis of the preservation of the primary constraints. Explicitly from (2.3) we find

$$\begin{aligned}\partial_t \pi_N &= \{\pi_N, H\} = -\mathcal{H}_T - \sqrt{g}\Omega \equiv -\mathcal{H}'_T \approx 0 , \\ \partial_t p_\Omega &= \{p_\Omega, H\} = -(N\sqrt{g} - 1) \equiv -\Gamma \approx 0 , \\ \partial_t \pi_i &= \{\pi_i, H\} = -\mathcal{H}_i \approx 0 .\end{aligned}\tag{2.10}$$

Then the total Hamiltonian with all constraints included has the form

$$H_T = \int d^3\mathbf{x} (N\mathcal{H}_T + v_T\mathcal{H}'_T + (v_\Gamma + \Omega)\Gamma + N^i\mathcal{H}_i + v_\Omega p_\Omega + v_N\pi_N) ,\tag{2.11}$$

where N^i can now be considered as Lagrange multipliers corresponding to the constraints \mathcal{H}_i while we still keep N as dynamical variable while v_T and v_Ω are the Lagrange multipliers corresponding to the constraints \mathcal{H}'_T and Γ respectively.

Now we proceed to the analysis of the stability of all constraints. Using (2.11) we find

$$\partial_t p_\Omega = \{p_\Omega, H_T\} = -\Gamma - v_T\sqrt{g} \approx -v_T\sqrt{g}\tag{2.12}$$

that implies $v_T = 0$. In case of the constraint $\pi_N \approx 0$ we have

$$\partial_t \pi_N = \{\pi_N, H_T\} = -\mathcal{H}_T - (v_\Gamma + \Omega)\sqrt{g} = -\mathcal{H}'_T - v_\Gamma\sqrt{g} = 0\tag{2.13}$$

that again implies that $v_\Gamma = 0$. Let us now consider the time evolution of the constraint Γ

$$\begin{aligned}\partial_t \Gamma &= \{\Gamma, H_T\} = \\ &\equiv -8\pi G N^2 \pi^{ij} g_{ij} + \partial_i N^i N \sqrt{g} + v_N \sqrt{g} = 0\end{aligned}\tag{2.14}$$

that can be considered as the equation for the Lagrange multiplier v_N . In case of the constraint \mathcal{H}'_T we find

$$\partial_t \mathcal{H}'_T = \{\mathcal{H}'_T, H_T\} = \int d^3\mathbf{x} (\{\mathcal{H}'_T, N\mathcal{H}'_T\} + \sqrt{g}v_\Omega \approx \sqrt{g}v_\Omega = 0 ,\tag{2.15}$$

where in the first step we used (2.9). Then (2.15) implies $v_\Omega = 0$. Finally we consider the time evolution of the constraint \mathcal{H}_i . Due to the fact that Ω and N are dynamical variables it is natural to extend the constraint \mathcal{H}_i with the appropriate combination of the primary constraints p_Ω and π_N so that

$$\tilde{\mathcal{H}}_i = \mathcal{H}_i + p_\Omega \partial_i \Omega + \pi_N \partial_i N , \mathbf{T}_S(N^i) = \int d^3\mathbf{x} N^i \tilde{\mathcal{H}}_i .\tag{2.16}$$

Now the time evolution of the smeared form of the constraint $\mathbf{T}_S(M^i)$ is equal to

$$\begin{aligned}\partial_t \mathbf{T}_S(M^i) &= \{\mathbf{T}_S(M^i), H_T\} \approx \left\{ \mathbf{T}_S(M^i), \int d^3\mathbf{x} N \mathcal{H}'_T \right\} - \{\mathbf{T}_S(M^i), \Omega\} \approx \\ &\approx M^i \partial_i \Omega = 0 ,\end{aligned}\tag{2.17}$$

where we again used (2.9). Since the equation above has to be valid for all M^i we see that it corresponds to some form of the constraint on Ω . In order to explicitly identify the nature of given constraint we split Ω into the zero mode part and the remaining part as follows

$$\Omega(\mathbf{x}, t) = \Omega_0(t) + \bar{\Omega}(\mathbf{x}, t) , \quad \Omega(t) = \frac{1}{\int d^3\mathbf{x} \sqrt{g}} \int d^3\mathbf{x} \sqrt{g} \Omega(\mathbf{x}, t) , \tag{2.18}$$

where by definition $\int d^3\mathbf{x} \sqrt{g} \bar{\Omega}(\mathbf{x}, t) = 0$. Then the equation (2.17) implies

$$\bar{\Omega}(\mathbf{x}, t) = K(t) , \tag{2.19}$$

where from definition of $\bar{\Omega}$ we obtain

$$\int d^3\mathbf{x} \sqrt{g} \bar{\Omega}(\mathbf{x}, t) = K(t) \int d^3\mathbf{x} \sqrt{g} = 0 \tag{2.20}$$

and hence we find $K(t) = 0$. In other words we have following constraint

$$\bar{\Omega}(\mathbf{x}, t) = 0 \tag{2.21}$$

while the zero mode $\Omega_0(t)$ is still non-specified. It is useful to perform the similar separation of the zero mode part of p_Ω as well

$$\begin{aligned}p_\Omega(\mathbf{x}, t) &= \frac{\sqrt{g}}{\int d^3\mathbf{x} \sqrt{g}} P_\Omega(t) + \bar{p}_\Omega(\mathbf{x}, t) , \\ p_\Omega(t) &= \int d^3\mathbf{x} p_\Omega(\mathbf{x}, t) , \quad \int d^3\mathbf{x} \bar{p}_\Omega(\mathbf{x}, t) = 0 .\end{aligned}\tag{2.22}$$

Note that we included the factor $\frac{\sqrt{g}}{\int d^3\mathbf{x} \sqrt{g}}$ in front of p_Ω in order to have canonical Poisson bracket

$$\{\Omega_0, P_\Omega\} = 1 \tag{2.23}$$

and also in order to ensure that p_Ω transforms as density since P_Ω is scalar. Then by definition we also find

$$\{\bar{\Omega}(\mathbf{x}), P_\Omega\} = 0 , \{\bar{p}_\Omega, \Omega_0\} = 0 . \tag{2.24}$$

It turns out that it is useful to perform similar separation in case of the constraint \mathcal{H}_T

$$\mathcal{H}_T = \frac{\sqrt{g}}{\int d^3\mathbf{x} \sqrt{g}} \mathcal{H}_0 + \bar{\mathcal{H}}_T , \quad \mathcal{H}_0 = \int d^3\mathbf{x} \mathcal{H}_T , \quad \int d^3\mathbf{x} \bar{\mathcal{H}}_T = 0 \tag{2.25}$$

and also in case of the Lagrange multiplier v_N

$$v_N = v_0^N + \bar{v}_N, \quad v_0^N = \frac{1}{\int d^3\mathbf{x}\sqrt{g}} \int d^3\mathbf{x}\sqrt{g}v_N, \quad \int d^3\mathbf{x}\sqrt{g}\bar{v}_N = 0. \quad (2.26)$$

Note that the Poisson brackets between $\bar{\mathcal{H}}_T$ still have the form as (2.9). Explicitly, let us define smeared form of this constraint

$$\bar{\mathbf{T}}_T(N) = \int d^3\mathbf{x}N(\mathbf{x})\bar{\mathcal{H}}_T(\mathbf{x}) = \int d^3\mathbf{x}\bar{N}(\mathbf{x})\bar{\mathcal{H}}_T(\mathbf{x}), \quad (2.27)$$

where we performed the separation $N = N_0 + \bar{N}$, $\int d^3\mathbf{x}N\sqrt{g} = 0$. Then we have

$$\{\bar{\mathbf{T}}_T(N), \bar{\mathbf{T}}(M)\} = \{\mathbf{T}_T(\bar{N}), \mathbf{T}_T(\bar{M})\} = \int d^3\mathbf{x}((\bar{N}\partial_i\bar{M} - \partial_i\bar{N}\bar{M})g^{ij}\mathcal{H}_j) \quad (2.28)$$

and hence the right side vanishes on the constraint surface $\mathcal{H}_i \approx 0$. With the help of the separation (2.25) and (2.26) we find the total Hamiltonian in the form

$$H_T = \int d^3\mathbf{x}(N\mathcal{H}_T + \bar{v}_N\bar{\mathcal{H}}_T + v_0^N \int d^3\mathbf{x}\sqrt{g}\Phi + N^i\tilde{\mathcal{H}}_i + (v_\Gamma + \Omega_0)\Gamma + v_\Omega P_\Omega + v_N\pi_N), \quad (2.29)$$

where

$$\Phi \equiv \frac{1}{\int d^3\mathbf{x}\sqrt{g}}\mathcal{H}_0 + \Omega_0 \approx 0, \quad (2.30)$$

and where we do not consider the modes $\bar{\Lambda} \approx 0, \bar{\rho}_\Omega \approx 0$ that are canonically conjugate the second class constraints that decouple from the theory. Before we proceed further we should also modify the constraint $\bar{\mathcal{H}}_T$ and $\tilde{\mathcal{H}}_i$ in such a way that they Poisson commute with Γ and Φ . In fact, let us consider following modification of the constraint $\bar{\mathcal{H}}_T$

$$\bar{\mathcal{H}}'_T = \bar{\mathcal{H}}_T + \frac{1}{32\pi Gg}g^{ij}\pi_{ij}\pi_N. \quad (2.31)$$

Now it is easy to see that

$$\{\bar{\mathcal{H}}'_T(\mathbf{x}), \Gamma(\mathbf{y})\} = 0. \quad (2.32)$$

Further we have to ensure that $\bar{\mathcal{H}}'_T$ Poisson commute with the constraint Φ . Clearly we have $\{\bar{\mathcal{H}}_T, H_0\} \approx 0$, while

$$\left\{\bar{\mathbf{T}}_T(N), \frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right\} = 8\pi G \left(\frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right)^2 \int d^3\mathbf{x}\bar{N}\pi^{ij}g_{ij} \quad (2.33)$$

so that in order to cancel this contribution we extend the constraint $\bar{\mathcal{H}}'_T$ so that it has the form

$$\bar{\mathcal{H}}''_T = \bar{\mathcal{H}}'_T + 8\pi G \left(\frac{1}{\int d^3\mathbf{x}\sqrt{g}}\right)^2 \overline{\pi^{ij}g_{ij}}P_\Omega, \quad (2.34)$$

where by definition $\int d^3\mathbf{x}\overline{\pi^{ij}g_{ij}} = 0$. In the similar way we modify the diffeomorphism constraint $\tilde{\mathcal{H}}_i$ so that it Poisson commute with Γ (Note that it has vanishing Poisson bracket with Φ on the constraint surface automatically)

$$\bar{\mathcal{H}}_i = \tilde{\mathcal{H}}_i + \partial_i \left[\frac{\pi_N}{\sqrt{g}} \right]. \quad (2.35)$$

so that

$$\{\mathbf{T}_S(N^i), \Gamma(\mathbf{y})\} = -N^k \partial_k \Gamma - \partial_i N^i \Gamma \approx 0. \quad (2.36)$$

In summary we have following total Hamiltonian

$$H_T = \int d^3\mathbf{x} (N\mathcal{H}_T + \bar{v}_N \bar{\mathcal{H}}_T'' + v_0^N \int d^3\mathbf{x} \sqrt{g} \Phi + N^i \bar{\mathcal{H}}_i + (v_\Gamma + \Omega_0) \Gamma + v_\Omega P_\Omega + v_N \pi_N) \quad (2.37)$$

and check stability of all constraints:

$$\partial_t \pi_N = \{\pi_N, H_T\} = -\mathcal{H}_T - \Omega_0 \sqrt{g} - v_\Gamma \sqrt{g} \approx -v_\Gamma \sqrt{g} = 0 \quad (2.38)$$

that implies that $v_\Gamma = 0$. For P_Ω we obtain

$$\partial_t P_\Omega = \{P_\Omega, H_T\} = -\Gamma - v_0^N \int d^3\mathbf{x} \sqrt{g} = 0 \quad (2.39)$$

that determines v_0^N to be equal to zero. For the constraint Γ we find

$$\partial_t \Gamma = \{\Gamma, H_T\} = v_N = 0 \quad (2.40)$$

and we find $v_N = 0$. Finally for Φ we obtain

$$\partial_t \Phi = \{\Phi, H_T\} = v_\Omega = 0 \quad (2.41)$$

and we again find $v_\Omega = 0$. Now we should proceed to the analysis of the time evolution of the constraints $\bar{\mathcal{H}}_i, \bar{\mathcal{H}}_T''$. However these constraints Poisson commute with the second class constraints by construction and also the Poisson brackets among themselves vanish on the constraint surface according to (2.9) and (2.28).

In summary we found that $P_\Omega \approx 0, \pi_N \approx 0, \Gamma \approx 0, \Phi \approx 0$ are the second class constraints. Solving these constraints we eliminate $N, \pi_N, \Omega_0, P_\Omega$ as functions of dynamical variables. Then the remaining constraints $\bar{\mathcal{H}}_T'', \bar{\mathcal{H}}_i$ form the set of $4\infty^3 - 1$ first class constraints with agreement with [18].

3. Unimodular Gravity in Henneaux-Teitelboim Form

In this section we consider the Henneaux-Teitelboim formulation of unimodular gravity that is based on the existence of the space-time vector density \mathcal{F}^μ . In this case the action has the form [17]

$$S = \frac{1}{16\pi G} \int d^4x [\sqrt{-\hat{g}} ((^4)R - 2\Lambda) + 2\Lambda \partial_\mu \mathcal{F}^\mu], \quad (3.1)$$

where $\Lambda(\mathbf{x}, t)$ is space-time dependent Lagrange multiplier. Our goal is to perform the canonical analysis of given theory. Firstly we find following collection of the primary constraints

$$\pi_N \approx 0, \quad \pi_i \approx 0, \quad \Gamma \equiv p_t^{\mathcal{F}} - \frac{1}{8\pi G} \Lambda \approx 0, \quad p_i^{\mathcal{F}} \approx 0, \quad p_\Lambda \approx 0, \quad (3.2)$$

where $p_t^{\mathcal{F}}, p_i^{\mathcal{F}}$ are momenta conjugate to $\mathcal{F}^t, \mathcal{F}^i$ respectively with following canonical Poisson brackets

$$\{\mathcal{F}^t(\mathbf{x}), p_t^{\mathcal{F}}(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}) , \{\mathcal{F}^i(\mathbf{x}), p_j^{\mathcal{F}}(\mathbf{y})\} = \delta_i^j \delta(\mathbf{x} - \mathbf{y}) . \quad (3.3)$$

Then we again find that the bare Hamiltonian with primary constraints included has the form

$$H = \int d^3\mathbf{x} \left(N (\mathcal{H}_T + p_t^{\mathcal{F}} \sqrt{g}) + N^i \mathcal{H}_i - \frac{1}{8\pi G} \Lambda \partial_i \mathcal{F}^i + v_N \pi_N + v^i \pi_i + v_\Lambda p_\Lambda + u_i p_i^{\mathcal{F}} + u_\Gamma \Gamma \right) , \quad (3.4)$$

where with the help of the constraint Γ we replaced $\frac{1}{8\pi G} \sqrt{g} \Lambda$ with $p_t^{\mathcal{F}} \sqrt{g}$. Now requirement of the preservation of the primary constraints imply following secondary constraints

$$\begin{aligned} \partial_t \pi_N &= \{\pi_N, H_T\} = -(\mathcal{H}_T + p_t^{\mathcal{F}} \sqrt{g}) \equiv -\mathcal{H}'_T \approx 0 , \\ \partial_t \pi_i &= \{\pi_i, H_T\} = -\mathcal{H}_i \approx 0 , \\ \partial_t \Gamma &= \{\Gamma, H_T\} = -\frac{v_\Lambda}{8\pi G} = 0 , \\ \partial_t p_i^{\mathcal{F}} &= \{p_i^{\mathcal{F}}, H_T\} = \frac{1}{8\pi G} \partial_i \Lambda , \\ \partial_t p_\Lambda &= \{p_\Lambda, H_T\} = \frac{1}{8\pi G} \partial_i \mathcal{F}^i + \frac{1}{8\pi G} u_\Gamma = 0 . \end{aligned} \quad (3.5)$$

The third and the fifth equation determines the Lagrange multipliers v_Λ and u_Γ . As in previous section we find that the fourth equation implies that $\bar{\Lambda}(t, \mathbf{x}) = 0$ while the zero mode part Λ_0 is not determined. In other words we have the second class constraints $\bar{\Lambda} = 0, p_{\bar{\Lambda}} = 0$ so we will not consider these modes anywhere and restrict ourselves to the case of the zero mode of Λ . Finally we modify \mathcal{H}_i in order to incorporate the transformation rule for Λ

$$\mathcal{H}'_i = \mathcal{H}_i + p_\Lambda \partial_i \Lambda \quad (3.6)$$

so that the total Hamiltonian has the form

$$H_T = \int d^3\mathbf{x} (N \mathcal{H}'_T + N^i \mathcal{H}'_i + v_\Lambda p_\Lambda + u_\Gamma \Gamma) , \quad (3.7)$$

where we also used integration by parts that eliminates the term $\Lambda \partial_i \mathcal{F}^i$. Finally we see that p_Λ and Γ are the second class constraint so that we can eliminate p_Λ and Λ from the theory. As a result we find the theory with $4\infty^3$ the first class constraints $\mathcal{H}'_T, \mathcal{H}'_i$ for the dynamical variables $g_{ij}, \pi^{ij}, p_t^{\mathcal{F}}, \mathcal{F}^t$. Note that the Hamiltonian does not depend on \mathcal{F}^t explicitly and hence we see that $p_t^{\mathcal{F}}$ is constant on-shell. In other words $p_t^{\mathcal{F}}$ plays the role of the cosmological constant which however is not included into the theory by hand but it arises as a consequence of the dynamics of the unimodular theory in Henneaux-Teitelboim formulation.

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